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Monotonicity properties of the superposition operator and their applications

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Abstract

We establish some properties of the superposition operator which are associated with monotonicity. Those properties are expressed in terms of the notion of degree of decrease or degree of increase. An application of the obtained results to the study of solvability of a quadratic Volterra integral equation is also derived.

Keywords: Superposition operator; Degree of decrease; Measure of noncompactness; Quadratic integral equation; Fixed point theorem

1. Introduction

The main goal of this paper is to discuss the properties of the so-called superposition operator related to monotonicity. More precisely, we introduce an index measuring the degree of improvement of monotonicity of functions transformed by the superposition operator.

The results concerning monotonicity properties of the superposition operator will be applied in the investigations of the solvability of a nonlinear quadratic integral equation of Volterra type. Namely, we are going to show that under some assumptions the mentioned integral equation has monotonic and nonnegative (or positive) solutions in the space of real functions continuous on some bounded and closed interval.

The superposition operator is one of the simplest nonlinear operator used in nonlinear functional analysis. On the other hand it is very important in the theory of integral and differential equations (cf. Ref. [2]).

In order to define this operator assume that J is a nonempty subset of real line $\mathbb{R}$. Consider the set X_J of real functions acting from the interval $[a,b]$ into the set J. Further, let $f:[a,b]\times J\to\mathbb{R}$ be a given function.

Then, to every function $x\in X_J$ we may assign the function Fx defined by the formula

$$(Fx)(t)=f\big(t,x(t)\big),\quad t\in[a,b].$$

The operator F defined in such a way is called *the superposition operator* generated by the function $f=f(t,x)$.

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The theory concerning the superposition operator is presented in Ref. [2].

The main tools used in our considerations are the concept of *a measure of noncompactness* and the concept of *degree of monotonicity* of a real function.

In order to present the first concept mentioned above suppose that E is a real Banach space with a norm $\|\cdot\|$. For a given nonempty subset X of E denote by $\overline{X}$, $\operatorname{Conv} X$ the closure and the closed convex hull of X, respectively.

Further, let $\mathfrak{M}_E$ denote the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

We accept the following definition from [4].

Definition. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0, +\infty)$ is said to be the measure of noncompactness in E, if it satisfies the following conditions:

(1) the family $\ker \mu = \{X \in \mathfrak{M}_E\colon\ \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$;
(2) $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$;
(3) $\mu(\overline{X}) = \mu(\operatorname{Conv} X) = \mu(X)$;
(4) $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$;
(5) if (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots,$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ described in (1) is called *the kernel of the measure of noncompactness* μ.

Further facts concerning measures of noncompactness and their properties may be found in [4]. For our purposes we will only need the following fixed point theorem [4,9].

Theorem 1. *Let Q be a nonempty bounded closed convex subset of the space E and let $T : Q \to Q$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\mu(TX) \leqslant k\mu(X)$ for any nonempty subset X of Q. Then T has a fixed point in the set Q.*

Remark 1. It can be shown that under the assumptions of the above theorem the set $\operatorname{Fix} T$ of fixed points of T belonging to Q is a member of the family $\ker \mu$ [4]. This fact allows us to characterize solutions of investigated equations.

In what follows let $I = [a, b]$ be a fixed interval in $\mathbb{R}$. Denote by $C = C(I)$ the classical Banach space of all real functions defined and continuous on I with the standard maximum norm $\|x\| = \max\{|x(t)|\colon\ t \in I\}$.

Now, let us fix a set $X \in \mathfrak{M}_C$. For $x \in X$ let us define the following quantities (cf. [6]):

$$d(x) = \sup\big\{\big|x(s) - x(t)\big| - \big[x(s) - x(t)\big]\colon\ t, s \in I,\ t \leqslant s\big\},$$
$$i(x) = \sup\big\{\big|x(t) - x(s)\big| - \big[x(t) - x(s)\big]\colon\ t, s \in I,\ t \leqslant s\big\}.$$

Analogously, put

$$d(X) = \sup\big\{d(x)\colon\ x \in X\big\},$$
$$i(X) = \sup\big\{i(x)\colon\ x \in X\big\}.$$

Observe that $d(x) = 0$ if and only if x is nondecreasing on I. Similarly, $d(X) = 0$ if and only if all functions belonging to X are nondecreasing on I. Thus the index $d(x)$ represents the degree of decrease of the function x on I. Analogously, the quantity $d(X)$ measures the degree of decrease of functions from the set X. In the same way we can characterize the quantity $i(x)$ and $i(X)$.

In what follows fix $\varepsilon > 0$ and denote by $\omega(x, \varepsilon)$ *the modulus of continuity* of the function x, i.e.

$$\omega(x, \varepsilon) = \sup\big\{\big|x(s) - x(t)\big|\colon\ t, s \in I,\ |t - s| \leqslant \varepsilon\big\}.$$

Similarly, let us put:

$$\omega(X, \varepsilon) = \sup\big\{\omega(x, \varepsilon)\colon\ x \in X\big\},$$
$$\omega_0(X) = \lim_{\varepsilon\to 0} \omega(X, \varepsilon).$$

Finally, we define the function μ on the family $\mathfrak{M}_C$ by putting

$$\mu(X) = \omega_0(X) + d(X). \tag{1.1}$$

It can be shown [6] that the function μ is a measure of noncompactness in the space $C(I)$ with the kernel $\ker\mu$ consisting of all nonempty and bounded sets X such that functions from X are equicontinuous and nondecreasing on I.

For other properties of μ we refer to [6].

Remark 2. In the same way as above we can define the measure of noncompactness associated with the quantity $i(X)$. We omit details.

2. Properties of the superposition operator related to monotonicity

In this section we will investigate the superposition operator F generated by a function $f = f(t,x)$ (cf. the previous section). We will assume that f is a real function defined on the set $I \times J$, where $I = [a,b]$ and J is an arbitrary real interval. We consider the superposition operator $(Fx)(t) = f(t, x(t))$ under the following assumptions:

(α) f is continuous on the set $I \times J$.
(β) The function $t \to f(t,x)$ is nondecreasing on I for any fixed $x \in J$.
(γ) For any fixed $t \in I$ the function $x \to f(t,x)$ is nondecreasing on J.
(δ) The function $f = f(t,x)$ satisfies the Lipschitz condition with respect to the variable x, i.e. there exists a constant $k > 0$ such that for any $t \in I$ and for $x_1, x_2 \in J$ the following inequality holds

$$\left|f(t,x_2) - f(t,x_1)\right| \leqslant k|x_2 - x_1|.$$

In what follows denote (similarly as earlier) by X_J the subset of $C(I)$ consisting of all functions $x : I \to J$.

Then we have the following result.

Theorem 2. *Assume that the hypotheses* (α)–(δ) *are satisfied and* $x \in X_J$. *Then*

$$d(Fx) \leqslant k d(x). \tag{2.1}$$

Proof. Denote by I_e the subset of $I \times I$ defined as follows

$$I_e = \{(t,s) \in I \times I\colon\ t < s \text{ and } x(t) = x(s)\}.$$

Obviously, for $(t,s) \in I_e$ we have:

$$\begin{aligned}
&\left|(Fx)(s) - (Fx)(t)\right| - \left[(Fx)(s) - (Fx)(t)\right] \\
&\quad = \left|f\big(s, x(s)\big) - f\big(t, x(t)\big)\right| - \left[f\big(s, x(s)\big) - f\big(t, x(t)\big)\right] \\
&\quad = \left|f\big(s, x(t)\big) - f\big(t, x(t)\big)\right| - \left[f\big(s, x(t)\big) - f\big(t, x(t)\big)\right] \\
&\quad = 0.
\end{aligned} \tag{2.2}$$

Now, assume that $t, s \in I$, $t < s$ and $(t,s) \notin I_e$, i.e $x(t) \neq x(s)$. Then we get:

$$\begin{aligned}
&\left|(Fx)(s) - (Fx)(t)\right| - \left[(Fx)(s) - (Fx)(t)\right] \\
&\quad = \left|f\big(s, x(s)\big) - f\big(t, x(t)\big)\right| - \left[f\big(s, x(s)\big) - f\big(t, x(t)\big)\right] \\
&\quad \leqslant \left|f\big(s, x(s)\big) - f\big(t, x(s)\big)\right| + \left|f\big(t, x(s)\big) - f\big(t, x(t)\big)\right| - \left[f\big(s, x(s)\big) - f\big(t, x(s)\big)\right] \\
&\qquad - \left[f\big(t, x(s)\big) - f\big(t, x(t)\big)\right] \\
&\quad = \left[f\big(s, x(s)\big) - f\big(t, x(s)\big)\right] + \left|f\big(t, x(s)\big) - f\big(t, x(t)\big)\right| - \left[f\big(s, x(s)\big) - f\big(t, x(s)\big)\right] \\
&\qquad - \left[f\big(t, x(s)\big) - f\big(t, x(t)\big)\right] \\
&\quad = \left|f\big(t, x(s)\big) - f\big(t, x(t)\big)\right| - \left[f\big(t, x(s)\big) - f\big(t, x(t)\big)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \left|\frac{f(t,x(s))-f(t,x(t))}{x(s)-x(t)}\right| \cdot \left|x(s)-x(t)\right| - \frac{f(t,x(s))-f(t,x(t))}{x(s)-x(t)}\left[x(s)-x(t)\right] \\
&= \left|\frac{f(t,x(s))-f(t,x(t))}{x(s)-x(t)}\right| \cdot \left|x(s)-x(t)\right| - \left|\frac{f(t,x(s))-f(t,x(t))}{x(s)-x(t)}\right| \cdot \left[x(s)-x(t)\right] \\
&= \frac{|f(t,x(s))-f(t,x(t))|}{|x(s)-x(t)|}\left\{\left|x(s)-x(t)\right| - \left[x(s)-x(t)\right]\right\} \\
&\leqslant \frac{k|x(s)-x(t)|}{|x(s)-x(t)|} \cdot \left\{\left|x(s)-x(t)\right| - \left[x(s)-x(t)\right]\right\} \\
&= k\left\{\left|x(s)-x(t)\right| - \left[x(s)-x(t)\right]\right\}. \qquad (2.3)
\end{aligned}
$$

Let us mention that in the above calculations we used the fact that the expression

$$\frac{f(t,x(s))-f(t,x(t))}{x(s)-x(t)}$$

is nonnegative.

Finally, let us observe that linking (2.2) and (2.3) we get the inequality (2.1).

Thus the proof is complete. □

From the above theorem follows that in the case when the function f satisfies the Lipschitz condition with a constant $k < 1$ (cf. the assumption (δ)) the superposition operator F generated by the function f improves the degree of monotonicity of any subset X of X_J with the coefficient k.

In the sequel we give a few convenient conditions being special cases of Theorem 1.

Corollary 1. *Suppose the function $f(t,x) = f : I \times J \to \mathbb{R}$ satisfies the assumptions (α), (β) of Theorem 2. Moreover, we assume that f has partial derivative f_x which is nonnegative and bounded on the set $I \times J$. Then f satisfies the assumptions (γ) and (δ) with the Lipschitz constant k defined as follows*

$$k = \sup\left\{f_x(t,x)\colon\ (t,x) \in I \times J\right\}.$$

Indeed, taking arbitrary $x_1, x_2 \in J$ and $t \in I$ and applying mean value theorem we infer that there exists $\theta \in (0,1)$ such that

$$f(t,x_2) - f(t,x_1) = f_x\big(t, x_1 + \theta(x_2 - x_1)\big)(x_2 - x_1).$$

Hence we deduce that the function $x \to f(t,x)$ is nondecreasing on J for any fixed $t \in I$ and satisfies the Lipschitz condition as in (δ).

Corollary 2. *Assume that the function $h : J \to \mathbb{R}$ is differentiable on the interval J and the derivative h' is nonnegative and bounded on J. Then the function $f(t,x) = h(x)$ satisfies the assumptions of Theorem 2 and the constant k appearing in (δ) is given by the equality $k = \sup\{h'(x)\colon\ x \in J\}$.*

Obviously this corollary is an easy consequence of Corollary 1.

Let us pay attention to the fact that in the paper [12] the authors assumed that $h : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable on $\mathbb{R}_+$ and the derivative h' is nonnegative and nondecreasing on $\mathbb{R}_+$.

Obviously such assumptions imply that the function h satisfies the assumptions of Corollary 2 for $J = [0, M]$, where $M > 0$ is arbitrarily fixed. Thus the result contained in [12] is a special case of that formulated in Corollary 2. Moreover, we show in the sequel that our result from Corollary 2 is more general than that from [12].

Now we illustrate our results by a few examples.

Example 1. Let $h : \mathbb{R}_+ \to \mathbb{R}$, $h(x) = \sqrt{x}$. Obviously $h'(x) = 1/2\sqrt{x}$ for $x > 0$. Thus on every interval of the form $J = [\alpha, +\infty)$ with $\alpha > 0$ the function h satisfies the assumptions of Corollary 2. Moreover, $k = \sup\{1/2\sqrt{x}\colon\ x \geqslant \alpha\} = 1/2\sqrt{\alpha}$.

On the other hand the function h does not satisfy the assumptions formulated in [12]. This shows that the result from Corollary 2 is more general than that contained in [12].

Example 2. Consider the function $f(t,x) = f : I \times [0,\alpha] \to \mathbb{R}$ defined by the formula $f(t,x) = \sin tx$, where α is a fixed number from the interval $(0, \pi/2)$ and $I = [0,1]$. Obviously the function $t \to f(t,x)$ is nondecreasing on I for any fixed $x \in [0,\alpha]$. On the other hand $f_x(t,x) = t\cos tx \geqslant 0$ for $(t,x) \in I \times [0,\alpha]$, thus $x \to f(t,x)$ is nondecreasing and f_x is bounded on $I \times [0,\alpha]$. Moreover, we have that $k = \sup\{f_x(t,x)\colon (t,x) \in I \times [0,\alpha]\} = 1$.

These facts show that $f(t,x)$ satisfies the assumptions of Corollary 1 with $k = 1$.

Example 3. Now, let us take into account the function $f(t,x) = f : I \times \mathbb{R}_+ \to \mathbb{R}$ ($I = [0,1]$) defined in the following way

$$f(t,x) = \begin{cases} 0 & \text{for } t = 0 \text{ and } x \in \mathbb{R}_+, \\ tx^2 & \text{for } t > 0 \text{ and } 0 \leqslant x \leqslant 1/\sqrt{t}, \\ 1 & \text{for } t > 0 \text{ and } x > 1\sqrt{t}. \end{cases}$$

It is not difficult to check that $f(t,x)$ satisfies the assumptions of Theorem 2 with the constant $k = 2$. On the other hand the partial derivative $f_x(t,x)$ does not exist on the curve $x = 1/\sqrt{t}$, so in this case we cannot apply Corollary 1.

3. Monotonic solutions of a quadratic integral equation

In this section we will consider the following nonlinear quadratic integral equation of Volterra type

$$x(t) = g(t) + f\big(t, x(t)\big) \int_0^t v\big(t, \tau, x(\tau)\big)\, d\tau, \tag{3.1}$$

where $t \in I = [0,1]$. Integral equations of such a type play very important role in nonlinear analysis and find numerous applications in engineering, mathematical physics, economics, biology and so on (cf. [1,7,8,10,11,13–15]).

Using the measure of noncompactness defined in Section 1 and the results established in Section 2 we show that Eq. (3.1) has monotonic and nonnegative solutions. The result we are going to prove here generalizes several ones obtained earlier in [1,3,5,10,12,14,15], for example.

Equation (3.1) will be studied under the following assumptions:

(i) $g \in C(I)$ and g is nondecreasing and nonnegative on the interval I.

(ii) The function $f : I \times J \to \mathbb{R}$ satisfies the conditions (α)–(γ) formulated in Section 2, where J is an unbounded interval such that $J \subset \mathbb{R}_+$ and $g_0 \in J$, where $g_0 = g(0) = \min\{g(t)\colon t \in I\}$. Moreover, f is nonnegative on $I \times J$.

(iii) There exists a nondecreasing function $k(r) = k : [g_0, +\infty) \to \mathbb{R}_+$ such that

$$\big|f(t,x_1) - f(t,x_2)\big| \leqslant k(r)|x_1 - x_2|$$

for any $t \in I$ and for all $x_1, x_2 \in [g_0, r]$.

(iv) $v : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $v : I \times I \times \mathbb{R}_+ \to \mathbb{R}_+$ and for arbitrarily fixed $\tau \in I$ and $x \in \mathbb{R}_+$ the function $t \to v(t,\tau,x)$ is nondecreasing on I.

(v) There exists a nondecreasing function $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$u(t,\tau,x) \leqslant p(x)$$

for $t, \tau \in I$ and $x \geqslant 0$.

(vi) There exists a positive solution r_0 of the inequality

$$\|g\| + \big(rk(r) + F_1\big)p(r) \leqslant r,$$

where $F_1 = \sup\{f(t,0)\colon t \in I\}$. Moreover, $k(r_0)p(r_0) < 1$.

Now, we can formulate our existence result.

Theorem 3. *Under the assumptions* (i)–(vi) *Eq.* (3.1) *has at least one solution $x = x(t)$ which belongs to the space $C(I)$ and is nondecreasing and nonnegative on the interval I.*

Proof. Let us take the subset S of the space $C(I)$, $S = \{x \in C(I)\colon\ x(t) \geqslant g_0 \text{ for } t \in I\}$. Consider the operator T defined on the set S by the formula

$$(Tx)(t) = g(t) + f\big(t, x(t)\big) \int_0^t v\big(t, \tau, x(\tau)\big)\, d\tau.$$

It is easily seen that our assumptions imply that the operator T transforms the set S into itself. Moreover, for an arbitrary $x \in S$ and $t \in I$ we get:

$$\begin{aligned}(Tx)(t) &\leqslant \|g\| + \big[\big|f\big(t, x(t)\big) - f(t,0)\big| + f(t,0)\big] \int_0^t p\big(x(\tau)\big)\, d\tau \\ &\leqslant \|g\| + \big(k\big(\|x\|\big)x(t) + F_1\big) \int_0^t p\big(\|x\|\big)\, d\tau \\ &\leqslant \|g\| + \big(\|x\| k\big(\|x\|\big) + F_1\big) p\big(\|x\|\big).\end{aligned}$$

Hence

$$\|Tx\| \leqslant \|g\| + \big(\|x\| k\big(\|x\|\big) + F_1\big) p\big(\|x\|\big).$$

Thus, taking into account the assumption (vi) we infer that there exists $r_0 > 0$ with $k(r_0)p(r_0) < 1$ such that the operator T transforms the set $S_{r_0} = \{x \in S\colon\ \|x\| \leqslant r_0\}$ into itself.

Let us mention that S_{r_0} is nonempty since $r_0 \geqslant g_0$. Moreover, S_{r_0} is bounded, closed and convex subset of $C(I)$.

Now, we show that T is continuous on the set S_{r_0}. To do this let us fix $\varepsilon > 0$ and take arbitrarily $x, y \in S_{r_0}$ such that $\|x - y\| \leqslant \varepsilon$. Then, for $t \in I$ we derive the following estimates:

$$\begin{aligned}\big|(Tx)(t) - (Ty)(t)\big| &\leqslant \left| f\big(t, x(t)\big) \int_0^t v\big(t, \tau, x(\tau)\big)\, d\tau - f\big(t, y(t)\big) \int_0^t v\big(t, \tau, y(\tau)\big)\, d\tau \right| \\ &\leqslant \left| f\big(t, x(t)\big) \int_0^t v\big(t, \tau, x(\tau)\big)\, d\tau - f\big(t, y(t)\big) \int_0^t v\big(t, \tau, x(\tau)\big)\, d\tau \right| \\ &\quad + \left| f\big(t, y(t)\big) \int_0^t v\big(t, \tau, x(\tau)\big)\, d\tau - f\big(t, y(t)\big) \int_0^t v\big(t, \tau, y(\tau)\big)\, d\tau \right| \\ &\leqslant k(r_0)\varepsilon_0 \int_0^t v\big(t, \tau, x(\tau)\big)\, d\tau + f\big(t, y(t)\big) \int_0^t \big|v\big(t, \tau, x(\tau)\big) - v\big(t, \tau, y(\tau)\big)\big|\, d\tau \\ &\leqslant p(r_0)k(r_0)\varepsilon + \big(r_0 k(r_0) + F_1\big)\omega^3_{r_0}(v, \varepsilon),\end{aligned}$$

where

$$\omega^3_{r_0}(v, \varepsilon) = \sup\big\{\big|v(t, \tau, x) - v(t, \tau, y)\big|\colon\ t, \tau \in I,\ x, y \in [g_0, r_0],\ |x - y| \leqslant \varepsilon\big\}.$$

Obviously $\omega^3_{r_0}(v, \varepsilon) \to 0$ as $\varepsilon \to 0$ which is a consequence of the uniform continuity of the function v on the set $I \times I \times [g_0, r_0]$.

From the above estimate we derive the following inequality

$$\|Tx - Ty\| \leqslant p(r_0)k(r_0)\varepsilon + \big(r_0 k(r_0) + F_1\big)\omega^3_{r_0}(v, \varepsilon)$$

which yields the continuity of the operator T on the set S_{r_0}.

Further, let us take a nonempty subset X of the set S_{r_0}. Fix arbitrarily a number $\varepsilon > 0$ and choose $x \in X$ and $t, s \in I$ such that $|t-s| \leqslant \varepsilon$. Without loss of generality we may assume that $t \leqslant s$. Then, in view of our assumptions, we obtain:

$$\begin{aligned}
\bigl|(Tx)(s)-(Tx)(t)\bigr| &\leqslant \bigl|g(s)-g(t)\bigr| + \left| f\bigl(s,x(s)\bigr)\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau - f\bigl(t,x(t)\bigr)\int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau \right| \\
&\leqslant \omega(g,\varepsilon) + \left| f\bigl(s,x(s)\bigr)\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau - f\bigl(t,x(t)\bigr)\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau \right| \\
&\quad + f\bigl(t,x(t)\bigr)\left| \int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau - \int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau \right| \\
&\leqslant \omega(g,\varepsilon) + \Bigl[\bigl|f\bigl(s,x(s)\bigr)-f\bigl(s,x(t)\bigr)\bigr| + \bigl|f\bigl(s,x(t)\bigr)-f\bigl(t,x(t)\bigr)\bigr|\Bigr]\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau \\
&\quad + \bigl(r_0k(r_0)+F_1\bigr)\left\{ \left| \int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau - \int_0^t v\bigl(s,\tau,x(\tau)\bigr)\,d\tau \right| \right. \\
&\quad \left. + \left| \int_0^t v\bigl(s,\tau,x(\tau)\bigr)\,d\tau - \int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau \right| \right\} \\
&\leqslant \omega(g,\varepsilon) + \bigl[k(r_0)\bigl|x(s)-x(t)\bigr| + \omega^1_{r_0}(f,\varepsilon)p(r_0)\bigr] \\
&\quad + \bigl(r_0k(r_0)+F_1\bigr)\left\{ \int_t^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau + \int_0^t \bigl|v\bigl(s,\tau,x(\tau)\bigr)-v\bigl(t,\tau,x(\tau)\bigr)\bigr|\,d\tau \right\} \\
&\leqslant \omega(g,\varepsilon) + \bigl[k(r_0)\omega(x,\varepsilon)+\omega^1_{r_0}(f,\varepsilon)\bigr]p(r_0) + \bigl(r_0k(r_0)+F_1\bigr)\bigl[\varepsilon p(r_0)+\omega^1_{r_0}(v,\varepsilon)\bigr],
\end{aligned}$$

where we denoted

$$\omega^1_{r_0}(f,\varepsilon) = \sup\bigl\{\bigl|f(s,x)-f(t,x)\bigr|\colon\ t,s \in I,\ x \in [g_0,r_0],\ |t-s| \leqslant \varepsilon\bigr\},$$
$$\omega^1_{r_0}(v,\varepsilon) = \sup\bigl\{\bigl|v(s,\tau,x)-v(t,\tau,x)\bigr|\colon\ t,s,\tau \in I,\ x \in [g_0,r_0],\ |s-t| \leqslant \varepsilon\bigr\}.$$

Notice that in view of the uniform continuity of the function f on the set $I \times [g_0,r_0]$ and the function v on the set $I \times I \times [g_0,r_0]$ we deduce that $\omega^1_{r_0}(f,\varepsilon) \to 0$ and $\omega^1_{r_0}(v,\varepsilon) \to 0$ as $\varepsilon \to 0$. This fact in conjunction with the above obtained estimate allows us to derive the following inequality

$$\omega_0(TX) \leqslant k(r_0)p(r_0)\omega_0(X). \tag{3.2}$$

Now, let us fix arbitrarily $t, s \in I$, $t \leqslant s$. For simplicity, denote by F the superposition operator generated by the function f (cf. Section 1). Then, for an arbitrary $x \in X$ we obtain:

$$\begin{aligned}
&\bigl|(Tx)(s)-(Tx)(t)\bigr| - \bigl[(Tx)(s)-(Tx)(t)\bigr] \\
&\quad = \left| a(s) + (Fx)(s)\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau - a(t) - (Fx)(t)\int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau \right| \\
&\qquad - \left[a(s) + (Fx)(s)\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau - a(t) - (Fx)(t)\int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau \right]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \bigl\{\bigl|a(s)-a(t)\bigr|-\bigl[a(s)-a(t)\bigr]\bigr\}+\left|(Fx)(s)\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau-(Fx)(t)\int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau\right| \\
&\quad-\left[(Fx)(s)\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau-(Fx)(t)\int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau\right] \\
&\leqslant \left|(Fx)(s)\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau-(Fx)(t)\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau\right| \\
&\quad+\left|(Fx)(t)\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau-(Fx)(t)\int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau\right| \\
&\quad-\left[(Fx)(s)\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau-(Fx)(t)\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau\right] \\
&\quad-\left[(Fx)(t)\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau-(Fx)(t)\int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau\right] \\
&\leqslant \bigl|(Fx)(s)-(Fx)(t)\bigr|\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau+(Fx)(t)\left|\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau-\int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau\right| \\
&\quad-\bigl[(Fx)(s)-(Fx)(t)\bigr]\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau-(Fx)(t)\left[\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau-\int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau\right] \\
&\leqslant \bigl\{\bigl|(Fx)(s)-(Fx)(t)\bigr|-\bigl[(Fx)(s)-(Fx)(t)\bigr]\bigr\}\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau \\
&\quad+(Fx)(t)\left|\int_0^t v\bigl(s,\tau,x(\tau)\bigr)\,d\tau+\int_t^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau-\int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau\right| \\
&\quad-(Fx)(t)\left[\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau-\int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau\right] \\
&\leqslant d(Fx)\int_0^s p(r_0)\,d\tau+(Fx)(t)\left\{\left|\int_0^t\bigl[v\bigl(s,\tau,x(\tau)\bigr)-v\bigl(t,\tau,x(\tau)\bigr)\bigr]\,d\tau\right|+\left|\int_t^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau\right|\right\} \\
&\quad-(Fx)(t)\left[\int_0^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau-\int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau\right].
\end{aligned}
$$

Hence, applying Theorem 2 and keeping in mind the assumption (iv), we get

$$
\begin{aligned}
&\bigl|(Tx)(s)-(Tx)(t)\bigr|-\bigl[(Tx)(s)-(Tx)(t)\bigr] \\
&\quad\leqslant k(r_0)p(r_0)\,d(x)+(Fx)(t)\left[\int_0^t v\bigl(s,\tau,x(\tau)\bigr)\,d\tau+\int_t^s v\bigl(s,\tau,x(\tau)\bigr)\,d\tau-\int_0^t v\bigl(t,\tau,x(\tau)\bigr)\,d\tau\right]
\end{aligned}
$$

$$- (Fx)(t)\left[\int_0^s v\big(s,\tau,x(\tau)\big)\,d\tau - \int_0^t v\big(t,\tau,x(\tau)\big)\,d\tau\right]$$
$$= k(r_0)p(r_0)\,d(x).$$

This estimate implies

$$d(Tx) \leqslant k(r_0)p(r_0)\,d(x)$$

and consequently

$$d(TX) \leqslant k(r_0)p(r_0)\,d(X). \tag{3.3}$$

Finally, joining the estimates (3.2) and (3.3) and taking into account the measure of noncompactness μ defined by (1.1), we obtain

$$\mu(TX) \leqslant k(r_0)p(r_0)\mu(X).$$

Now, keeping in mind the above inequality and the fact that $k(r_0)p(r_0) < 1$ (cf. the assumption (vi)), in view of Theorem 1 we complete the proof. □

Remark 3. Taking into account the description of the kernel of the measure of noncompactness μ (cf. Section 1) and Remark 1, we deduce easily that all solutions of Eq. (3.1) belonging to S_{r_0} are nondecreasing, nonnegative and continuous on the interval I. Obviously those solutions are positive provided $g_0 > 0$.

Remark 4. It is easily seen that our result is valid if we replace in our considerations the interval $I = [0,1]$ by an arbitrary interval $I = [a,b]$. In such a case Eq. (3.1) will have the form

$$x(t) = g(t) + f\big(t,x(t)\big)\int_a^t v\big(t,\tau,x(\tau)\big)\,d\tau, \quad t \in I = [a,b].$$

Obviously in this case we have to modify the assumption (vi). We omit the details.

In what follows we illustrate our results by a few examples.

Example 4. Consider the following quadratic integral equation

$$x(t) = \sin(t - 1 + \pi/2) + \sqrt{x(t)}\int_0^t \frac{(t^2+\tau^2)x(\tau)}{1+x^3(\tau)}\,d\tau, \tag{3.4}$$

for $t \in I = [0,1]$.

Observe that this equation is a special case of Eq. (3.1), where $g(t) = \sin(t-1+\pi/2)$, $f(t,x) = \sqrt{x}$, $v(t,\tau,x) = (t^2+\tau^2)x/(1+x^3)$. It is easily seen that there are satisfied the assumptions of Theorem 3.

Indeed, the function $g(t)$ is positive, nondecreasing on I and $g_0 = g(0) = \sin(-1+\pi/2) = 0.5356\ldots$, $\|g\| = 1$. The function $f(t,x) = \sqrt{x}$ is nondecreasing on $\mathbb{R}_+$ and for any $r \geqslant g_0$ we have that $k(r) = 1/2\sqrt{g_0} = 0.68319\ldots$ (cf. the assumption (iii) and Example 1). Further notice that the function $t \to v(t,\tau,x)$ is nondecreasing on I for fixed $\tau \in I$ and $x \geqslant 0$. Moreover, we have

$$v(t,\tau,x) \leqslant \frac{2x}{1+x^3} \leqslant 2^{5/3}/3 = 1.05826\ldots.$$

Thus we conclude that the function $p(x)$ appearing in the assumption (v) is constant, i.e. $p(x) = 2^{5/3}/3$. Taking into account that $F_1 = 0$ we obtain that the inequality from the assumption (vi) has the form

$$1 + r(1/2\sqrt{g_0})2^{5/3}/3 \leqslant r.$$

It is easy to check that the number.

$$r_0 = 1/\big[1 - (1/2\sqrt{g_0})2^{5/3}/3\big] = 3.610\ldots$$

is the solution of the above inequality for which $k(r_0)p(r_0) < 1$.

Now, from Theorem 3 we deduce that Eq. (3.4) has a positive, nondecreasing solution $x = x(t)$ being continuous on I and such that $x(t) \in [g_0, r_0]$ for $t \in I$.

Example 5. Now, we examine the quadratic integral equation having the form

$$x(t) = t^2 e^{-2t} + \big[t/(t+1)\big] \ln\big(1 + x(t)\big) \int_0^t \big(t\tau + x^2(\tau)\big)\, d\tau, \tag{3.5}$$

for $t \in I = [0, 1]$.

In this case we have $g(t) = t^2 e^{-2t}$, $f(t, x) = [t/(t+1)] \ln(1+x)$, $v(t, \tau, x) = t\tau + x^2$. The function g is nondecreasing and nonnegative on I and $g_0 = 0$, $\|g\| = 1/e^2$. The function f is nonnegative and nondecreasing with respect to both variables on I. Further, we have $f_x(t, x) = [t/(t+1)]/(1+x)$. This yields that f_x is nonincreasing on $\mathbb{R}_+$ for any $t \in I$, so in the light of Corollary 1 we have that $k(r) = 1/2$ for any $r > 0$. Moreover, the function $t \to v(t, \tau, x)$ is nondecreasing on I and we have $v(t, \tau, x) \leqslant 1 + x^2$ for $t, \tau \in I$ and $x \geqslant 0$. Thus $p(x) = 1 + x^2$.

Finally, consider the inequality from the assumption (vi) which has the form

$$1/e^2 + \frac{1}{2} r\big(1 + r^2\big) \leqslant r.$$

It is easy to check that $r_0 = 2/3$ is the solution of this inequality for which $k(r_0)p(r_0) = 13/18 < 1$.

Thus, in the light of Theorem 3 we infer that Eq. (3.5) has a nondecreasing, continuous and nonnegative solution $x = x(t)$ for $t \in I$.

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